

The Neutrino Mass Matrix – From A_4 to Z_3

Ernest Ma

Physics Department, University of California, Riverside, California 92521

Abstract

With the recent experimental advance in our precise knowledge of the neutrino oscillation parameters, the correct form of the 3×3 neutrino mass matrix is now approximately known. I discuss how this may be obtained from symmetry principles, using as examples the finite groups A_4 and Z_3 , in two complete theories of leptons (and quarks).

Preprint version of talk at the 9th Adriatic Meeting in Dubrovnik (2003).

1 Introduction

After the new experimental results of KamLAND [1] on top of those of SNO [2] and SuperKamiokande [3], etc. [4], we now have very good knowledge of 5 parameters:

$$\Delta m_{atm}^2 \simeq 2.5 \times 10^{-3} \text{ eV}^2, \quad (1)$$

$$\Delta m_{sol}^2 \simeq 6.9 \times 10^{-5} \text{ eV}^2, \quad (2)$$

$$\sin^2 2\theta_{atm} \simeq 1, \quad (3)$$

$$\tan^2 \theta_{sol} \simeq 0.46, \quad (4)$$

$$|U_{e3}| < 0.16. \quad (5)$$

The last 3 numbers tell us that the neutrino mixing matrix is rather well-known, and to a very good first approximation, it is given by

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} c & -s & 0 \\ s/\sqrt{2} & c/\sqrt{2} & -1/\sqrt{2} \\ s/\sqrt{2} & c/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \quad (6)$$

where $\sin^2 2\theta_{atm} = 1$ and $U_{e3} = 0$ have been assumed, with $s \equiv \sin \theta_{sol}$, $c \equiv \cos \theta_{sol}$.

2 Approximate Generic Form of the Neutrino Mass Matrix

Assuming three Majorana neutrino mass eigenstates with real eigenvalues $m_{1,2,3}$, the neutrino mass matrix in the basis $(\nu_e, \nu_\mu, \nu_\tau)$ is then of the form [5]

$$\mathcal{M}_\nu = \begin{pmatrix} a + 2b + 2c & d & d \\ d & b & a + b \\ d & a + b & b \end{pmatrix}. \quad (7)$$

Note that \mathcal{M}_ν is invariant under the discrete Z_2 symmetry: $\nu_e \rightarrow \nu_e$, $\nu_\mu \leftrightarrow \nu_\tau$. Depending on the relative magnitudes of the 4 parameters a, b, c, d , this matrix has 7 possible limits:

3 have the normal hierarchy, 2 have the inverted hierarchy, and 2 have 3 nearly degenerate masses.

In neutrinoless double beta decay, the effective mass is $m_0 = |a + 2b + 2c|$. In the 2 cases of inverted hierarchy, we have

$$m_0 \simeq \sqrt{\Delta m_{atm}^2} \simeq 0.05 \text{ eV}, \quad (8)$$

$$m_0 \simeq \cos 2\theta_{sol} \sqrt{\Delta m_{atm}^2}, \quad (9)$$

respectively for $m_1/m_2 = \pm 1$, i.e. for their relative CP being even or odd. In the 2 degenerate cases,

$$m_0 \simeq |m_{1,2,3}|, \quad (10)$$

$$m_0 \simeq \cos 2\theta_{sol} |m_{1,2,3}|. \quad (11)$$

With \mathcal{M}_ν of Eq. (7), U_{e3} is zero necessarily, in which case there can be no CP violation in neutrino oscillations. However, suppose we consider instead [5, 6]

$$\mathcal{M}_\nu = \begin{pmatrix} a + 2b + 2c & d & d^* \\ d & b & a + b \\ d^* & a + b & b \end{pmatrix}, \quad (12)$$

where d is now complex, then the Z_2 symmetry of Eq. (7) is broken and U_{e3} becomes nonzero. In fact, it is proportional to $iImd$, thus predicting maximal CP violation in neutrino oscillations.

3 Nearly Degenerate Majorana Neutrino Masses

Suppose that at some high energy scale, the charged lepton mass matrix and the Majorana neutrino mass matrix are such that after diagonalizing the former, i.e.

$$\mathcal{M}_l = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}, \quad (13)$$

the latter is of the form

$$\mathcal{M}_\nu = \begin{pmatrix} m_0 & 0 & 0 \\ 0 & 0 & m_0 \\ 0 & m_0 & 0 \end{pmatrix}. \quad (14)$$

From the high scale to the electroweak scale, one-loop radiative corrections will change \mathcal{M}_ν as follows:

$$(\mathcal{M}_\nu)_{ij} \rightarrow (\mathcal{M}_\nu)_{ij} + R_{ik}(\mathcal{M}_\nu)_{kj} + (\mathcal{M}_\nu)_{ik}R_{kj}^T, \quad (15)$$

where the radiative correction matrix is assumed to be of the most general form, i.e.

$$R = \begin{pmatrix} r_{ee} & r_{e\mu} & r_{e\tau} \\ r_{e\mu}^* & r_{\mu\mu} & r_{\mu\tau} \\ r_{e\tau}^* & r_{\mu\tau}^* & r_{\tau\tau} \end{pmatrix}. \quad (16)$$

Thus the observed neutrino mass matrix is given by

$$\mathcal{M}_\nu = m_0 \begin{pmatrix} 1 + 2r_{ee} & r_{e\tau} + r_{e\mu}^* & r_{e\mu} + r_{e\tau}^* \\ r_{e\mu}^* + r_{e\tau} & 2r_{\mu\tau} & 1 + r_{\mu\mu} + r_{\tau\tau} \\ r_{e\tau}^* + r_{e\mu} & 1 + r_{\mu\mu} + r_{\tau\tau} & 2r_{\mu\tau}^* \end{pmatrix}. \quad (17)$$

Let us rephase ν_μ and ν_τ to make $r_{\mu\tau}$ real, then the above \mathcal{M}_ν is exactly in the form of Eq. (12), with of course a as the dominant term. In other words, we have obtained a desirable description of all present data on neutrino oscillations including CP violation, starting from almost nothing.

4 Plato's Fire

The successful derivation of Eq. (17) depends on having Eqs. (13) and (14). To be sensible theoretically, they should be maintained by a symmetry. At first sight, it appears impossible that there can be a symmetry which allows them to coexist. The solution turns out to be the non-Abelian discrete symmetry A_4 [7, 8]. What is A_4 and why is it special?

Around the year 390 BCE, the Greek mathematician Theaetetus proved that there are five and only five perfect geometric solids. The Greeks already knew that there are four basic

elements: fire, air, water, and earth. Plato could not resist matching them to the five perfect geometric solids and for that to work, he invented the fifth element, i.e. quintessence, which is supposed to hold the cosmos together. His assignments are shown in Table 1.

Table 1: Properties of Perfect Geometric Solids

solid	faces	vertices	Plato	Group
tetrahedron	4	4	fire	A_4
octahedron	8	6	air	S_4
icosahedron	20	12	water	A_5
hexahedron	6	8	earth	S_4
dodecahedron	12	20	?	A_5

The group theory of these solids was established in the early 19th century. Since a cube (hexahedron) can be imbedded perfectly inside an octahedron and the latter inside the former, they have the same symmetry group. The same holds for the icosahedron and dodecahedron. The tetrahedron (Plato’s “fire”) is special because it is self-dual. It has the symmetry group A_4 , i.e. the finite group of the even permutation of 4 objects. The reason that it is special for the neutrino mass matrix is because it has three inequivalent one-dimensional irreducible representations and one three-dimensional irreducible representation exactly. Its character table is given below.

Table 2: Character Table of A_4

class	n	h	χ_1	χ_2	χ_3	χ_4
C_1	1	1	1	1	1	3
C_2	4	3	1	ω	ω^2	0
C_3	4	3	1	ω^2	ω	0
C_4	3	2	1	1	1	-1

In the above, n is the number of elements, h is the order of each element, and

$$\omega = e^{2\pi i/3} \quad (18)$$

is the cube root of unity. The group multiplication rule is

$$\underline{3} \times \underline{3} = \underline{1} + \underline{1}' + \underline{1}'' + \underline{3} + \underline{3}. \quad (19)$$

5 Details of the A_4 Model

The fact that A_4 has three inequivalent one-dimensional representations $\underline{1}$, $\underline{1}'$, $\underline{1}''$, and one three-dimensional representation $\underline{3}$, with the decomposition given by Eq. (19) leads naturally to the following assignments of quarks and leptons:

$$(u_i, d_i)_L, \quad (\nu_i, e_i)_L \sim \underline{3}, \quad (20)$$

$$u_{1R}, \quad d_{1R}, \quad e_{1R} \sim \underline{1}, \quad (21)$$

$$u_{2R}, \quad d_{2R}, \quad e_{2R} \sim \underline{1}', \quad (22)$$

$$u_{3R}, \quad d_{3R}, \quad e_{3R} \sim \underline{1}''. \quad (23)$$

Heavy fermion singlets are then added:

$$U_{iL(R)}, \quad D_{iL(R)}, \quad E_{iL(R)}, \quad N_{iR} \sim \underline{3}, \quad (24)$$

together with the usual Higgs doublet and new heavy singlets:

$$(\phi^+, \phi^0) \sim \underline{1}, \quad \chi_i^0 \sim \underline{3}. \quad (25)$$

With this structure, charged leptons acquire an effective Yukawa coupling matrix $\bar{e}_{iL} e_{jR} \phi^0$ which has 3 arbitrary eigenvalues (because of the 3 independent couplings to the 3 inequivalent one-dimensional representations) and for the case of equal vacuum expectation values of χ_i , i.e.

$$\langle \chi_1 \rangle = \langle \chi_2 \rangle = \langle \chi_3 \rangle = u, \quad (26)$$

which occurs naturally in the supersymmetric version of this model [8], the unitary transformation U_L which diagonalizes \mathcal{M}_l is given by

$$U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}. \quad (27)$$

This implies that the effective neutrino mass operator, i.e. $\nu_i \nu_j \phi^0 \phi^0$, is proportional to

$$U_L^T U_L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (28)$$

exactly as desired.

6 New Flavor-Changing Radiative Mechanism

The original A_4 model [7] was conceived to be a symmetry at the electroweak scale, in which case the splitting of the neutrino mass degeneracy is put in by hand and any mixing matrix is possible. Subsequently, it was proposed [8] as a symmetry at a high scale, in which case the mixing matrix is determined completely by flavor-changing radiative corrections and the only possible result happens to be Eq. (17). This is a remarkable convergence in that Eq. (17) is in the form of Eq. (12), i.e. the phenomenologically preferred neutrino mixing matrix based on the most recent data from neutrino oscillations.

We should now consider the new physics responsible for the r_{ij} 's of Eq. (16). Previously [8], arbitrary soft supersymmetry breaking in the scalar sector was invoked. It is certainly a phenomenologically viable scenario, but lacks theoretical motivation and is somewhat complicated. Here a new and much simpler mechanism is proposed [9], using a triplet of charged scalars under A_4 , i.e. $\eta_i^+ \sim \underline{3}$. Their relevant contributions to the Lagrangian of this model is then

$$\mathcal{L} = f \epsilon_{ijk} (\nu_i e_j - e_i \nu_j) \eta_k^+ + m_{ij}^2 \eta_i^+ \eta_j^-. \quad (29)$$

Whereas the first term is invariant under A_4 as it should be, the second term is a soft term which is allowed to break A_4 , from which the flavor-changing radiative corrections will be calculated.

Let

$$\begin{pmatrix} \eta_e \\ \eta_\mu \\ \eta_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}, \quad (30)$$

where $\eta_{1,2,3}$ are mass eigenstates with masses $m_{1,2,3}$. The resulting radiative corrections are given by

$$r_{\alpha\beta} = -\frac{f^2}{8\pi^2} \sum_{i=1}^3 U_{\alpha i}^* U_{\beta i} \ln m_i^2. \quad (31)$$

To the extent that $r_{\mu\tau}$ should not be larger than about 10^{-2} , the common mass m_0 of the three degenerate neutrinos should not be less than about 0.2 eV in this model. This is consistent with the recent WMAP upper bound [10] of 0.23 eV and the range 0.11 to 0.56 eV indicated by neutrinoless double beta decay [11].

7 Models based on S_3 and D_4

Two other examples of the application of non-Abelian discrete symmetries to the neutrino mass matrix have recently been proposed. One [12] is based on the symmetry group of the equilateral triangle S_3 , which has 6 elements and the irreducible representations $\underline{1}$, $\underline{1}'$, and $\underline{2}$. The 3 families of leptons as well as 3 Higgs doublets transform as $\underline{1} + \underline{2}$ under S_3 . An additional Z_2 is introduced where $\nu_R(\underline{1})$ and $H(\underline{2})$ are odd, while all other fields are even. After a detailed analysis, the mixing matrix of Eq. (6) is obtained with $U_{e3} \simeq -3.4 \times 10^{-3}$ and $0.4 < \tan \theta_{sol} < 0.8$. The neutrino masses are predicted to have an inverted hierarchy satisfying Eq. (8).

Another example [13] is based on the symmetry group of the square D_4 , which has 8 elements and the irreducible representations $\underline{1}^{++}$, $\underline{1}^{+-}$, $\underline{1}^{-+}$, $\underline{1}^{--}$, and $\underline{2}$. The 3 families of

leptons transform as $\underline{1}^{++} + \underline{2}$. The Higgs sector has 3 doublets with $\phi_3 \sim \underline{1}^{+-}$ and 2 singlets $\chi \sim \underline{2}$. Under an extra Z_2 , ν_R , e_R , ϕ_1 are odd, while all other fields are even, including ϕ_2 . This results in the neutrino mass matrix of Eq. (7) with an additional constraint, i.e. $m_1 < m_2 < m_3$ such that the m_0 of neutrinoless double beta decay is equal to $m_1 m_2 / m_3$.

8 Form Invariance of the Neutrino Mass Matrix

Consider a specific 3×3 unitary matrix U and impose the condition [14]

$$U \mathcal{M}_\nu U^T = \mathcal{M}_\nu \quad (32)$$

on the neutrino mass matrix \mathcal{M}_ν in the $(\nu_e, \nu_\mu, \nu_\tau)$ basis. Iteration of the above yields

$$U^n \mathcal{M}_\nu (U^T)^n = \mathcal{M}_\nu. \quad (33)$$

Therefore, unless $U^{\bar{n}} = 1$ for some finite \bar{n} , the only solution for \mathcal{M}_ν would be a multiple of the identity matrix. Take for example $\bar{n} = 2$, then the choice

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (34)$$

leads to Eq. (7). In other words, the present neutrino oscillation data may be understood as a manifestation of the discrete symmetry $\nu_e \rightarrow \nu_e$ and $\nu_\mu \leftrightarrow \nu_\tau$.

Suppose instead that $\bar{n} = 4$, with U^2 given by Eq. (34), then one possible solution for its square root is

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1-i)/\sqrt{2} & (1+i)/\sqrt{2} \\ 0 & (1+i)/\sqrt{2} & (1-i)/\sqrt{2} \end{pmatrix}, \quad (35)$$

which leads to

$$\mathcal{M}_1 = \begin{pmatrix} 2b+2c & d & d \\ d & b & b \\ d & b & b \end{pmatrix}, \quad (36)$$

i.e. the 4 parameters of Eq. (7) have been reduced to 3 by setting $a = 0$.

Another solution is

$$U_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad (37)$$

which leads to

$$\mathcal{M}_2 = \begin{pmatrix} 2b + 2d & d & d \\ d & b & b \\ d & b & b \end{pmatrix}, \quad (38)$$

i.e. \mathcal{M}_1 has been reduced by setting $c = d$. The 3 mass eigenvalues are then $m_{1,2} = 2b \mp \sqrt{2}d$ and $m_3 = 0$, i.e. an inverted hierarchy, with $\tan^2 \theta_{sol}$ predicted to be $2 - \sqrt{3} = 0.27$, as compared to the allowed range [15] 0.29 to 0.86 from fitting all present data.

9 New Z_3 Model of Neutrino Masses

Very recently, two new complete models of lepton masses have been obtained, one based on Z_4 [16] and the other on Z_3 [17]. The former does not fix the solar mixing angle, whereas the latter predicts $\tan^2 \theta_{sol} = 0.5$. Here I will discuss only the Z_3 case. Let \mathcal{M}_ν be given by

$$\mathcal{M}_\nu = \mathcal{M}_A + \mathcal{M}_B + \mathcal{M}_C, \quad (39)$$

where

$$\mathcal{M}_A = A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{M}_B = B \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{M}_C = C \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (40)$$

Since the invariance of \mathcal{M}_A requires only $U_A U_A^T = 1$, U_A can be any orthogonal matrix. As for \mathcal{M}_B and \mathcal{M}_C , they are both invariant under the Z_2 transformation of Eq. (34) and each is invariant under a Z_3 transformation, i.e. $U_B^3 = 1$ and $U_C^3 = 1$, but $U_B \neq U_C$. Specifically,

$$U_B = \begin{pmatrix} -1/2 & -\sqrt{3}/8 & -\sqrt{3}/8 \\ \sqrt{3}/8 & 1/4 & -3/4 \\ \sqrt{3}/8 & -3/4 & 1/4 \end{pmatrix}, \quad U_C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (41)$$

Note that U_B commutes with U_2 , but U_C does not. If U_C is combined with U_2 , then the non-Abelian discrete symmetry S_3 is generated.

First consider $C = 0$. Then $\mathcal{M}_\nu = \mathcal{M}_A + \mathcal{M}_B$ is the most general solution of

$$U_B \mathcal{M}_\nu U_B^T = \mathcal{M}_\nu, \quad (42)$$

and the eigenvectors of \mathcal{M}_ν are ν_e , $(\nu_\mu + \nu_\tau)/\sqrt{2}$, and $(\nu_\mu - \nu_\tau)/\sqrt{2}$ with eigenvalues $A - B$, $A - B$, and $A + B$ respectively. This explains atmospheric neutrino oscillations with $\sin^2 2\theta_{atm} = 1$ and

$$(\Delta m^2)_{atm} = (A + B)^2 - (A - B)^2 = 4BA. \quad (43)$$

Now consider $C \neq 0$. Then in the basis spanned by ν_e , $(\nu_\mu + \nu_\tau)/\sqrt{2}$, and $(\nu_\mu - \nu_\tau)/\sqrt{2}$,

$$\mathcal{M}_\nu = \begin{pmatrix} A - B + C & \sqrt{2}C & 0 \\ \sqrt{2}C & A - B + 2C & 0 \\ 0 & 0 & A + B \end{pmatrix}. \quad (44)$$

The eigenvectors and eigenvalues become

$$\nu_1 = \frac{1}{\sqrt{6}}(2\nu_e - \nu_\mu - \nu_\tau), \quad m_1 = A - B, \quad (45)$$

$$\nu_2 = \frac{1}{\sqrt{3}}(\nu_e + \nu_\mu + \nu_\tau), \quad m_2 = A - B + 3C, \quad (46)$$

$$\nu_3 = \frac{1}{\sqrt{2}}(\nu_\mu - \nu_\tau), \quad m_3 = A + B. \quad (47)$$

This explains solar neutrino oscillations as well with $\tan^2 \theta_{sol} = 1/2$ and

$$(\Delta m^2)_{sol} = (A - B + 3C)^2 - (A - B)^2 = 3C(2A - 2B + 3C). \quad (48)$$

Whereas the mixing angles are fixed, the proposed \mathcal{M}_ν has the flexibility to accommodate the three patterns of neutrino masses often mentioned, i.e.

- (I) the hierarchical solution, e.g. $B = A$ and $C \ll A$;
- (II) the inverted hierarchical solution, e.g. $B = -A$ and $C \ll A$;

(III) the degenerate solution, e.g. $C \ll B \ll A$.

In all cases, C must be small. Therefore \mathcal{M}_ν of Eq. (39) satisfies Eq. (42) to a very good approximation, and $Z_2 \times Z_3$ as generated by U_2 and U_B should be taken as the underlying symmetry of this model.

Since \mathcal{M}_C is small and breaks the symmetry of $\mathcal{M}_A + \mathcal{M}_B$, it is natural to think of its origin in terms of the well-known dimension-five operator [18]

$$\mathcal{L}_{eff} = \frac{f_{ij}}{2\Lambda}(\nu_i\phi^0 - l_i\phi^+)(\nu_j\phi^0 - l_j\phi^+) + H.c., \quad (49)$$

where (ϕ^+, ϕ^0) is the usual Higgs doublet of the Standard Model and Λ is a very high scale. As ϕ^0 picks up a nonzero vacuum expectation value v , neutrino masses are generated, and if $f_{ij}v^2/\Lambda = C$ for all i, j , \mathcal{M}_C is obtained. Since Λ is presumably of order 10^{16} to 10^{18} GeV, C is of order 10^{-3} to 10^{-5} eV, and $A - B + 3C/2$ is of order 10^{-2} to 1 eV. This range of values is just right to encompass all three solutions mentioned above.

To justify the assumption that U_B operates in the basis $(\nu_e, \nu_\mu, \nu_\tau)$, the complete theory of leptons must be discussed. Under the assumed Z_3 symmetry, the leptons transform as follows:

$$(\nu, l)_i \rightarrow (U_B)_{ij}(\nu, l)_j, \quad l_k^c \rightarrow l_k^c, \quad (50)$$

implemented by 3 Higgs doublets and 1 Higgs triplet:

$$(\phi^0, \phi^-)_i \rightarrow (U_B)_{ij}(\phi^0, \phi^-)_j, \quad (\xi^{++}, \xi^+, \xi^0) \rightarrow (\xi^{++}, \xi^+, \xi^0). \quad (51)$$

The Yukawa interactions of this model are then given by

$$\begin{aligned} \mathcal{L}_Y &= h_{ij}[\xi^0\nu_i\nu_j - \xi^+(\nu_i l_j + l_i \nu_j)/\sqrt{2} + \xi^{++}l_i l_j] \\ &+ f_{ij}^k(l_i\phi_j^0 - \nu_i\phi_j^-)l_k^c + H.c. \end{aligned} \quad (52)$$

with

$$h = \begin{pmatrix} a-b & 0 & 0 \\ 0 & a & -b \\ 0 & -b & a \end{pmatrix}, \quad \mathcal{M}_\nu = 2h\langle\xi^0\rangle, \quad (53)$$

and

$$f^k = \begin{pmatrix} a_k - b_k & d_k & d_k \\ -d_k & a_k & -b_k \\ -d_k & -b_k & a_k \end{pmatrix}. \quad (54)$$

Note that the d terms are absent in h because it has to be symmetric. Assume $v_{1,2} \ll v_3$, and $d_k \ll b_k \ll a_k$, then $V_L \mathcal{M}_l \mathcal{M}_l^\dagger V_L^\dagger = \text{diagonal}$ implies that V_L is nearly diagonal. This justifies the original choice of basis for \mathcal{M}_ν .

Any model of neutrino mixing implies the presence of lepton flavor violation at some level. In this case, ϕ_1^0 couples dominantly to $e\tau^c$ and ϕ_2^0 to $\mu\tau^c$. Taking into account also the other couplings, the branching fractions for $\mu \rightarrow eee$ and $\mu \rightarrow e\gamma$ are estimated to be of order 10^{-12} and 10^{-11} respectively for a Higgs mass of 100 GeV. Both are at the level of present experimental upper bounds.

10 Conclusions

The correct form of \mathcal{M}_ν is now approximately known. In the $(\nu_e, \nu_\mu, \nu_\tau)$ basis, it obeys the discrete symmetry of Eq. (34). Using Eq. (32), the phenomenologically successful Eq. (7) is obtained, which has 7 possible limits for \mathcal{M}_ν .

Assuming some additional symmetry, such as A_4 or Z_3 , with possible flavor changing radiative corrections, complete theories of leptons (and quarks) may be constructed with the prediction of specific neutrino mass patterns and other experimentally verifiable consequences.

Acknowledgements

I thank Josip Trampetic, Silvio Pallua, and the other organizers of Adriatic 2003 for their great hospitality at Dubrovnik. This work was supported in part by the U. S. Department

of Energy under Grant No. DE-FG03-94ER40837.

Appendix

It is amusing to note the parallel between the 5 perfect geometric solids and the 5 anomaly-free superstring theories in 10 dimensions. Whereas the former are related among themselves by geometric dualities, the latter are related by S, T, U dualities: Type I \leftrightarrow SO(32), Type IIa \leftrightarrow $E_8 \times E_8$, and Type IIb is self-dual. Whereas the 5 geometric solids may be embedded in a sphere, the 5 superstring theories are believed to be different limits of a single underlying M Theory.

References

- [1] K. Eguchi *et al.*, KamLAND Collaboration, Phys. Rev. Lett. **90**, 021802 (2003).
- [2] Q. R. Ahmad *et al.*, SNO Collaboration, Phys. Rev. Lett. **89**, 011301, 011302 (2002).
- [3] For a recent review, see for example C. K. Jung, C. McGrew, T. Kajita, and T. Mann, Ann. Rev. Nucl. Part. Sci. **51**, 451 (2001).
- [4] M. Apollonio *et al.*, Phys. Lett. B 466, 415 (1999); F. Boehm *et al.*, Phys. Rev. D 64, 112001 (2001).
- [5] E. Ma, Phys. Rev. **D66**, 117301 (2002).
- [6] W. Grimus and L. Lavoura, hep-ph/0305309.
- [7] E. Ma and G. Rajasekaran, Phys. Rev. D 64, 113012 (2001); E. Ma, Mod. Phys. Lett. A 17, 289; 627 (2002).
- [8] K. S. Babu, E. Ma, and J. W. F. Valle, Phys. Lett. **B552**, 207 (2003).

- [9] E. Ma, Mod. Phys. Lett. **A17**, 2361 (2002).
- [10] D. N. Spergel *et al.*, Astrophys. J. Suppl. **148**, 175 (2003).
- [11] H. V. Klapdor-Kleingrothaus *et al.*, Mod. Phys. Lett. A **16**, 2409 (2001).
- [12] J. Kubo, A. Mondragon, M. Mondragon, and E. Rodriguez-Jauregui, Prog. Theor. Phys. **109**, 795 (2003).
- [13] W. Grimus and L. Lavoura, Phys. Lett. **B572**, 189 (2003).
- [14] E. Ma, Phys. Rev. Lett. **90**, 221802 (2003).
- [15] M. Maltoni, T. Schwetz, and J. W. F. Valle, Phys. Rev. **D67**, 0903003 (2003).
- [16] E. Ma and G. Rajasekaran, Phys. Rev. **D68**, 071302(R) (2003).
- [17] E. Ma, hep-ph/0308282.
- [18] S. Weinberg, Phys. Rev. Lett. **43**, 1566 (1979).